

# Induced gravity in $\mathbb{Z}_N$ orientifold models\*

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## Abstract

We consider non-compact  $\mathbb{Z}_N$  orientifold models of type IIB superstring theory with four-dimensional gravity induced on a set of coincident D3-branes. For the models with odd  $N$  the contribution to the one-loop renormalization of the Planck mass is shown to come only from the torus and to be  $O(N)$  as the contributions from annulus, Moebius strip and Klein bottle cancel. One can therefore realize the Dvali-Gabadadze-Porrati idea that four-dimensional gravity is induced by quantum effects at the one-loop level by considering large  $N$ .

## 1 Introduction

Today we know of several ways to realize four-dimensional gravity (see e.g. [1] for a review). First, we get a four-dimensional Einstein-Hilbert term from a  $D$ -dimensional one if we compactify  $D - 4$  dimensions. Second, we can have a cosmological constant in the bulk and on a codimension one 3-brane. This is the Randall-Sundrum mechanism [2] and [3]. Third, we can have brane induced gravity, i.e. we can have a four-dimensional Einstein-Hilbert term that is induced on a 3-brane by the fields that live on the 3-brane. This has been put forward by Dvali, Gabadadze and Porrati ([4] – [6]). Obviously we can also combine these three ideas in a given model.

Both Randall-Sundrum and Dvali-Gabadadze-Porrati originally work in a five-dimensional bulk (i.e. with codimension one). Whereas the Randall-Sundrum setup leads to a four-dimensional behaviour of gravity at long distances and a five-dimensional behaviour of gravity at short distances (that is similar to compactification) in the case of the Dvali-Gabadadze-Porrati

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setup gravity is four-dimensional at short distances and five-dimensional at long distances. Therefore to be consistent with experiments and astronomical observations the cross-over scale has either to be astronomically large or one has to compactify the extra dimension [7]. In the later case Kaluza-Klein graviton emission is suppressed in the ultraviolet and therefore also the energy loss from brane to bulk. Constraints from experiments on the size of the compact dimension are therefore less stringent than for standard compactifications.

With a codimension greater than one the effects of a finite brane thickness can no longer be neglected ([8] – [12]), but the conclusions on the cross-over scale or on the need for compactification remain the same. On the other hand one can also merge the setups of Randall-Sundrum and of Dvali-Gabadadze-Porrati [13].

In a complete theory we need four-dimensional gravity and we will consider how this can be achieved in superstring theory. We may ask the question if there are string models with brane induced gravity where the one-loop induced four-dimensional Planck mass is large in string units. For heterotic string theory such corrections vanish for  $\mathcal{N} \geq 1$  supersymmetry ([14] – [16]). For type II vacua such corrections can be non-vanishing for  $\mathcal{N} \leq 2$  supersymmetry [14],[17]. In particular for a background of the type  $M_4 \times CY_3$ , where  $M_4$  is four-dimensional Minkowski space and  $CY_3$  is a Calabi-Yau, the one-loop correction is proportional to the Euler number ([17] – [19]).

In this paper we will consider type I/orientifold vacua. These models can have gauge and matter fields on the D3-branes that come close to the standard model (and supersymmetric generalizations thereof). Explicitly we will consider models that are non-compact (non-standard) orientifolds of type IIB superstring theory on symmetric  $\mathbb{Z}_N$  orbifolds and we will compute the induced gravity on a set of coincident D3-branes. The reasons are the following: As in this setup we have the gauge fields, matter fields and gravity localized on the D3-branes we do not need to compactify. Besides being interesting for its own, this has the advantage that we can have arbitrary  $N$  and may consider the large  $N$  limit and that we can have an arbitrary number of D3-branes (no untwisted tadpole cancellation condition).

Our orientifold models are similar to the ones discussed in [20] and [21] but more general as we consider models with arbitrary possibly large  $N$ . In [22] the anomaly cancellation in these models has been analysed. We will determine the renormalization of the four-dimensional Planck mass. Whereas in [20] and [21] the contributions to the Planck mass are only stated to exist and to be determined by the string scale we explicitly determine them by a string computation and show that they are large if  $N$  is large. This is a

generalization of the computation of [23] and [19] that considered compactification on K3. We can then compare the torus versus the annulus, Moebius strip and Klein bottle contributions.

Writing the one-loop renormalization of the four-dimensional Planck mass as

$$\Delta\mathcal{L}_{\text{eff}}^{1\text{-loop}} = \delta M_s^2 \sqrt{-g} R \quad (1)$$

we will show that the torus contribution is  $O(N)$  and that annulus, Moebius strip and Klein bottle contributions cancel. Therefore, by considering large  $N$  the one-loop contribution can be arbitrary large. The number of gauge and matter fields on the D3-branes is not growing with  $N$  and we can in principle find models that are quite close to supersymmetric generalizations of the standard model.

There is also a different string theory realization of induced gravity presented in [8] that is based on the orientifold of K3 and two extra compactified dimensions as in [23]. There the contribution to the four-dimensional Planck mass comes only from the Kaluza-Klein tower from annulus, Moebius strip and Klein bottle as the torus does not contribute.

In section 2 we consider  $\mathbb{Z}_N$  orbifolds of type IIB and review the contribution to the Planck mass from the torus. In sections 3 we consider  $\mathbb{Z}_N$  orientifolds of type IIB and compute the contribution to the Planck mass from the annulus, the Moebius strip and the Klein bottle. In section 4 we give our conclusions. Some details of the computations are left to appendices.

## 2 $\mathbb{Z}_N$ orbifolds

In order to have localized twisted sectors and therefore a localized Einstein term and in order to have a parameter  $N$  that we may e.g. assume to be large we consider  $\mathbb{Z}_N$  orbifolds of type IIB superstring theory in this section. In the compact case we compactify on  $M^4 \times T^6/\mathbb{Z}_N$ , where  $M^4$  is four-dimensional Minkowski space. In the non-compact case the background is  $M^4 \times \mathbb{R}^6/\mathbb{Z}_N$ . We have  $\mathcal{N} = 2$  supersymmetry in  $d = 4$ . In section 2.1 we will first review how the contribution of the torus to the one-loop renormalization of the Planck mass is determined by the Euler number or the second helicity supertrace. In section 2.2 we then show how the second helicity supertrace follows from the helicity generation partition function. In section 2.3 we analyse the large  $N$  limit. In 2.4 we compute the torus contribution from a two graviton amplitude in order to fix the vertex operator normalization that we will need in section 3.

## 2.1 The second helicity supertrace

The helicity supertraces are defined by (see [16])

$$B_{2n} = \text{Str} [\lambda^{2n}] \quad n \in \mathbb{N}, \quad (2)$$

where the  $\lambda$ 's are the helicity eigenvalues. The contribution of the  $\mathcal{N} = 2$  supergravity multiplet and of  $\mathcal{N} = 2$  vector multiplets to the second helicity supertrace  $B_2$  is 1, whereas the contribution of  $\mathcal{N} = 2$  hyper multiplets is  $-1$ . This gives

$$B_2 = 1 + n_V - n_H, \quad (3)$$

where  $n_V$  and  $n_H$  count the number of vector and hyper multiplets. The  $\mathbb{Z}_N$  orbifolds we consider are singular limits of Calabi-Yau 3-folds with hodge numbers

$$h^{1,1} = n_V, \quad h^{2,1} = n_H - 1, \quad (4)$$

where we have subtracted the universal hyper multiplet. The Euler number is

$$\chi = 2(h^{1,1} - h^{2,1}). \quad (5)$$

This gives

$$B_2 = \frac{1}{2}\chi. \quad (6)$$

The only one-loop surface is the torus  $\mathcal{T}$ . In [17] it was shown (see also [19] and [24]) that this gives a one-loop renormalization of the Planck mass of<sup>1</sup>

$$\Delta \mathcal{L}_{\text{eff}}^{1\text{-loop}} = \delta_{\mathcal{T}} M_s^2 \sqrt{-g} R = \frac{1}{12\pi} \chi M_s^2 \sqrt{-g} R = \frac{1}{6\pi} B_2 M_s^2 \sqrt{-g} R. \quad (7)$$

With  $\lambda = \lambda_L + \lambda_R$  we get from the helicity generating partition function  $Z(v, \bar{v}) = \text{str}(q^{L_0} \bar{q}^{\bar{L}_0} \exp(2\pi i(v\lambda_R - \bar{v}\lambda_L)))$  the second helicity supertrace as (see [16])

$$B_2 = - \left( \frac{1}{2\pi i} \partial_v - \frac{1}{2\pi i} \partial_{\bar{v}} \right)^2 Z(v, \bar{v}) \Big|_{v=\bar{v}=0}. \quad (8)$$

## 2.2 The helicity generating partition function

Let us define the complex bosons as  $Z^i = X^{2i+2} + iX^{2i+3}$ ,  $i = 1, 2, 3$ . The helicity generating torus partition function for the  $\mathbb{Z}_N$  orbifold of type IIB is<sup>2</sup> (see also [18])

$$Z^{(0,0)}(v, \bar{v}) = N_0(N) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_X^2(\tau) \left[ Z_{\psi}^+(v, \tau) Z_{\psi}^+(v, \tau)^* \right] \Big|_{h=g=0} \prod_{i=1}^3 Z_i \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \quad (9)$$

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<sup>1</sup>We have  $M_s^2 = 1/\alpha'$ .

<sup>2</sup>In the non-compact case the partition function for the (0,0) sector is normalized with respect to the ten-dimensional volume and the partition function for the remaining sectors with respect to the four-dimensional volume. To derive the spectrum and the helicity supertraces one starts with the unintegrated partition function.

$$Z'(v, \bar{v}) = N_0(N) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_X^2(\tau) \sum_{\substack{h, g=0 \\ (h, g) \neq (0, 0)}}^{N-1} Z_\psi^+(v, \tau) Z_\psi^+(v, \tau)^* \prod_{i=1}^3 Z_i \left[ \begin{smallmatrix} hv_i \\ gv_i \end{smallmatrix} \right] (\tau), \quad (10)$$

where

$$Z_X^2(\tau) = \frac{1}{\tau_2} \frac{1}{|\eta(\tau)|^4} \quad (11)$$

$$Z_i \left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (\tau) = \frac{\Gamma_{2,2}}{|\eta(\tau)|^4} \quad (\Gamma_{2,2} \text{ is the } (2, 2) \text{ lattice sum}) \quad (12)$$

$$Z_i \left[ \begin{smallmatrix} hv_i \\ gv_i \end{smallmatrix} \right] (\tau) = C^{(N)}(hv_i, gv_i) \left| \frac{\eta(\tau)}{\theta \left[ \begin{smallmatrix} 1/2 + hv_i \\ 1/2 + gv_i \end{smallmatrix} \right] (0, \tau)} \right|^2 \quad \text{for } (hv_i, gv_i) \neq (0, 0) \quad (13)$$

$$Z_\psi^+(v, \tau) = \frac{\xi(v)}{2} \frac{1}{\eta(\tau)^4} \sum_{\alpha, \beta=0}^1 (-)^{\alpha+\beta+\alpha\beta} \theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 \end{smallmatrix} \right] (v, \tau) \theta \left[ \begin{smallmatrix} \alpha/2 + hv_1 \\ \beta/2 + gv_1 \end{smallmatrix} \right] (0, \tau) \\ \times \theta \left[ \begin{smallmatrix} \alpha/2 + hv_2 \\ \beta/2 + gv_2 \end{smallmatrix} \right] (0, \tau) \theta \left[ \begin{smallmatrix} \alpha/2 + hv_3 \\ \beta/2 + gv_3 \end{smallmatrix} \right] (0, \tau) \quad (14)$$

$$\xi(v) = \frac{\sin \pi v}{\pi} \frac{\partial_u \theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (u, \tau) \Big|_{u=0}}{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (v, \tau)}. \quad (15)$$

For the untwisted sector in the compact case

$$|C^{(N)}(0, gv_i)| = 4 (\sin(\pi gv_i))^2 \quad (16)$$

whereas in the non-compact case  $|C^{(N)}(0, gv_i)| = 1$  as there is an extra factor of  $1/(4 (\sin(\pi gv_i))^2)$  coming from the integration over non-compact momenta<sup>3</sup>. For the twisted sectors  $(h \neq 0)$   $|C^{(N)}(hv_i, gv_i)|$  counts the fixed points multiplicity that is always 1 in the non-compact case. The torus partition function

$$Z_{\mathcal{T}} = Z(0, 0) \quad (17)$$

is as expected modular invariant (use  $\xi(0) = 1$ ). Using the Riemann identity we get

$$Z_\psi^+(v, \tau) = \frac{\xi(v)}{\eta(\tau)^4} \theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (v/2, \tau) \theta \left[ \begin{smallmatrix} 1/2 - hv_1 \\ 1/2 - gv_1 \end{smallmatrix} \right] (v/2, \tau) \\ \times \theta \left[ \begin{smallmatrix} 1/2 - hv_2 \\ 1/2 - gv_2 \end{smallmatrix} \right] (v/2, \tau) \theta \left[ \begin{smallmatrix} 1/2 - hv_3 \\ 1/2 - gv_3 \end{smallmatrix} \right] (v/2, \tau). \quad (18)$$

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<sup>3</sup>The author thanks E. Dudas and P. Vanhove for clarifying this point

Let  $\mathcal{M}$  be the set of elements  $\{(h, g)\}$  that solve

$$hv_1 = gv_1 = 0 \pmod{1} \quad (19)$$

$$\text{or } hv_2 = gv_2 = 0 \pmod{1} \quad (20)$$

$$\text{or } hv_3 = gv_3 = 0 \pmod{1}. \quad (21)$$

Obviously  $(0, 0) \in \mathcal{M}$ . From (8) we get the second helicity supertrace for the  $\mathbb{Z}_N$  orbifold

$$B_2 = \frac{1}{2} N_0(N) \sum_{\substack{h, g = 0 \\ (h, g) \notin \mathcal{M}}}^{N-1} \left| C^{(N)}(hv_1, gv_1) C^{(N)}(hv_2, gv_2) C^{(N)}(hv_3, gv_3) \right|. \quad (22)$$

Notice that in the non-compact case only the twisted states ( $h \neq 0$ ) contribute as they are the ones that are localized in four-dimensions and induce the four-dimensional Planck mass. The normalization

$$N_0(N) = \frac{1}{N} \quad (23)$$

is fixed by matching the massless spectrum that one gets from the operator approach with the one one derives from the helicity generating partition function. In appendix B we first show this for the example of the  $\mathbb{Z}_3$  orbifold and then give the proof for prime  $N$ . The proof for the case with general  $N \in \mathbb{N}$  is only sketched as it is straight forward and lengthy.

### 2.3 Large $N$ behaviour of $\mathbb{Z}_N$ orbifolds

For a non-compact  $\mathbb{Z}_N$  orbifold the second helicity supertrace coming from twisted sectors is  $B_2^T = n_V^T - n_H^T$ . We have

$$N \text{ even: } n_V^T = \frac{N-2}{2} + 1 = \frac{N}{2}, \quad n_H^T = 0 \quad (24)$$

$$N \text{ odd: } n_V^T = \frac{N-1}{2}, \quad n_H^T = 0. \quad (25)$$

and therefore we have the behavior

$$B_2^T \xrightarrow{N \rightarrow \infty} \frac{N}{2} + O(1) \quad (26)$$

that gives using (7)

$$\Delta \mathcal{L}_{\text{eff}}^{1\text{-loop}} \xrightarrow{N \rightarrow \infty} \frac{N + O(1)}{12\pi} M_s^2 \sqrt{-g} R. \quad (27)$$

In the compact case the shift vector  $v$  for a  $\mathbb{Z}_N$  orbifold has to be such that the orbifold acts crystallographically. In the non-compact case this is obviously not necessary. For the  $\mathbb{Z}_N$  orbifold with odd  $N$  we can e.g. choose the shift vector  $v = (\frac{1}{N}, \frac{1}{N}, -\frac{2}{N})$ . Then  $\mathcal{M} = \{(0, 0)\}$  and (22) gives the twisted contribution (25).

## 2.4 The Planck mass from the two graviton amplitude

In this section we compute the torus contribution from a two graviton amplitude in order to fix the vertex operator normalization that we will need in section 3.

### 2.4.1 Matching amplitudes and effective actions

Let us define

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad (28)$$

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\lambda} (\partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda}) \quad (29)$$

$$R_{\nu\rho\sigma}^\mu = \partial_\rho \Gamma_{\nu\sigma}^\mu - \partial_\sigma \Gamma_{\nu\rho}^\mu + \Gamma_{\rho\lambda}^\mu \Gamma_{\nu\sigma}^\lambda - \Gamma_{\sigma\lambda}^\mu \Gamma_{\nu\rho}^\lambda. \quad (30)$$

With the graviton

$$h_{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \varepsilon_{\mu\nu}, \quad (31)$$

neglecting terms proportional to  $p_1^2, p_2^2, p_1 \cdot p_2, p_{1\mu} \varepsilon_1^{\mu\nu}, p_{2\rho} \varepsilon_2^{\rho\sigma}, \eta_{\mu\nu} \varepsilon_1^{\mu\nu}, \eta_{\rho\sigma} \varepsilon_2^{\rho\sigma}$  (due to the fact that gravitons are massless and that the polarization tensors are physical) and keeping the momenta arbitrary (no momentum conservation) we find in momentum space

$$\sqrt{-g}R|_{O(\kappa^2)} \rightarrow -\frac{\kappa^2}{8} \left( \eta_{\mu\rho} p_{1\sigma} p_{2\nu} + \eta_{\mu\sigma} p_{1\rho} p_{2\nu} + \eta_{\nu\rho} p_{1\sigma} p_{2\mu} + \eta_{\nu\sigma} p_{1\rho} p_{2\mu} \right) \varepsilon_1^{\mu\nu} \varepsilon_2^{\rho\sigma}. \quad (32)$$

It is enough to consider only one tensor structure as it follows from covariance that the only term in the effective action that contributes in second order in momentum is  $\sqrt{-g}R$ . Let us write the (off-shell) two graviton amplitude as

$$A^{(2)}|_{O(p^2)} = -\frac{1}{4C_m} \delta \left( \eta_{\mu\rho} p_{1\sigma} p_{2\nu} + \eta_{\mu\sigma} p_{1\rho} p_{2\nu} + \eta_{\nu\rho} p_{1\sigma} p_{2\mu} + \eta_{\nu\sigma} p_{1\rho} p_{2\mu} \right) \varepsilon_1^{\mu\nu} \varepsilon_2^{\rho\sigma}, \quad (33)$$

where the momenta are measured in string units (i.e. they are dimensionless) and we have introduced a matching coefficient  $C_m$  that we will determine and that accommodates the fact that we use vertex operators that are not normalized properly. Then the contribution to the effective action is precisely

$$\Delta\mathcal{L}_{\text{eff}} = M_s^2 \delta \sqrt{-g}R, \quad (34)$$

where a factor of  $\frac{1}{2}$  from Bose symmetry is taken into account.

### 2.4.2 The two graviton amplitude

The Einstein term is CP even and gets contributions from the even-even and the odd-odd spin structure two graviton amplitudes. Using the notation of

appendix C the even-even spin structure two graviton amplitude is

$$A_{(e-e)}^{(2)} = \sum_{(\alpha,\beta)=0,1}^{even} \sum_{(\bar{\alpha},\bar{\beta})=0,1}^{even} \int_{\Gamma} \frac{d^2\tau}{\tau_2^2} (-)^{\alpha+\beta+\alpha\beta} (-)^{\bar{\alpha}+\bar{\beta}+\bar{\alpha}\bar{\beta}} Z(\tau, \bar{\tau}, (\alpha, \beta), (\bar{\alpha}, \bar{\beta})) \\ \times \int d^2z_1 \int d^2z_2 \langle V^{(0,0)}(z_1, \bar{z}_1) V^{(0,0)}(z_2, \bar{z}_2) \rangle_{(\alpha,\beta),(\bar{\alpha},\bar{\beta})} \quad (35)$$

with the graviton vertex operator in the  $(0,0)$ -ghost picture

$$V^{(0,0)}(z, \bar{z}) = -\frac{2g_s}{\alpha'} \varepsilon_{\mu\nu} : \left( i\partial X^\mu - \frac{\alpha'}{2} \psi^\mu p \cdot \psi \right) \left( i\bar{\partial} X^\nu + \frac{\alpha'}{2} \tilde{\psi}^\nu p \cdot \tilde{\psi} \right) e^{ip \cdot X} : . \quad (36)$$

The piece in second order in momentum vanishes due to (166) and (167) and there are no pinching contributions from  $O(p^4)$ . The other possible contribution comes from the odd-odd spin structure two graviton amplitude

$$A_{(o-o)}^{(2)} = \int_{\Gamma} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}, (1, 1), (1, 1)) \int d^2z_1 \int d^2z_2 \\ \times \langle V^{(0,0)}(z_1, \bar{z}_1) V^{(-1,-1)}(z_2, \bar{z}_2) X^{pc}(z_{pc}, \bar{z}_{pc}) \rangle, \quad (37)$$

where the  $(-1, -1)$ -ghost picture vertex operator is

$$V^{(-1,-1)} = g_s \varepsilon_{\mu\nu} : \psi^\mu \tilde{\psi}^\nu e^{ip \cdot X} : \quad (38)$$

and the picture changing operator is

$$X^{pc} = \partial X^\alpha \psi_\alpha \bar{\partial} X^\beta \tilde{\psi}_\beta. \quad (39)$$

Actually to get the right tensor structure we have to consider a different distribution of the pictures. This is due to our choice of off-shell procedure and can be avoided by considering amplitudes with more gravitons <sup>4</sup>. We start with

$$A_{(o-o)}^{(2)} = \int_{\Gamma} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}, (1, 1), (1, 1)) \int d^2z_1 \int d^2z_2 \\ \times \langle V^{(0,-1)}(z_1, \bar{z}_1) V^{(-1,0)}(z_2, \bar{z}_2) X^{pc}(z_{pc}, \bar{z}_{pc}) \rangle. \quad (40)$$

The amplitude is independent on the position of the picture changing operator  $z_{pc}$ . We have 4 fermion zero modes and the first non-vanishing correlator has 4 fermions

$$\langle \psi^\mu(z_1) \psi^\nu(z_1) \psi^\rho(z_2) \psi^\sigma(z_{pc}) \rangle = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{\alpha'} g_1(z_1, z_2, z_{pc}, \tau) \\ \langle \tilde{\psi}^\mu(\bar{z}_1) \tilde{\psi}^\nu(\bar{z}_2) \tilde{\psi}^\rho(\bar{z}_2) \tilde{\psi}^\sigma(\bar{z}_{pc}) \rangle = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{\alpha'} g_2(z_1, z_2, z_{pc}, \tau)^*. \quad (41)$$

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<sup>4</sup>The author thanks P. Vanhove for clarifying this point.



Using

$$\varepsilon_{\alpha}^{\mu\gamma\rho}\varepsilon^{\alpha\nu\delta\sigma} = -\det \begin{pmatrix} \eta^{\mu\nu} & \eta^{\mu\delta} & \eta^{\mu\sigma} \\ \eta^{\gamma\nu} & \eta^{\gamma\delta} & \eta^{\gamma\sigma} \\ \eta^{\rho\nu} & \eta^{\rho\delta} & \eta^{\rho\sigma} \end{pmatrix} \quad (42)$$

and neglecting terms proportional to  $p_1^2, p_2^2, p_1 \cdot p_2, p_{1\mu}\varepsilon_1^{\mu\nu}, p_{2\rho}\varepsilon_2^{\rho\sigma}, \eta_{\mu\nu}\varepsilon_1^{\mu\nu}, \eta_{\rho\sigma}\varepsilon_2^{\rho\sigma}$  (what leaves only one term from (42)) we find the piece in second order in momentum

$$\begin{aligned} A_{(o-o)}^{(2)} \Big|_{O(p^2)} &= g_s^2 \int_{\Gamma} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}, (1,1), (1,1)) \frac{1}{\alpha'^2} h(\tau, \bar{\tau}) \\ &\times \frac{\alpha'}{8} \left( \eta_{\mu\rho} p_{1\sigma} p_{2\nu} + \eta_{\mu\sigma} p_{1\rho} p_{2\nu} + \eta_{\nu\rho} p_{1\sigma} p_{2\mu} + \eta_{\nu\sigma} p_{1\rho} p_{2\mu} \right) \varepsilon_1^{\mu\nu} \varepsilon_2^{\rho\sigma}, \end{aligned} \quad (43)$$

where

$$h(\tau, \bar{\tau}) = \int d^2 z_1 \int d^2 z_2 g_1 g_2^* \langle \partial X(z_{pc}, \bar{z}_{pc}) \bar{\partial} X(z_{pc}, \bar{z}_{pc}) \rangle. \quad (44)$$

From now on we measure positions and momenta in string units ( $\frac{1}{\alpha'} d^2 z \rightarrow d^2 z, \alpha' p_{\mu} p_{\nu} \rightarrow p_{\mu} p_{\nu}$ ). Comparing with (33) we get

$$\delta_{\mathcal{T}} = -\frac{1}{2} g_s^2 C_m \int_{\Gamma} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}, (1,1), (1,1)) h(\tau, \bar{\tau}). \quad (45)$$

The odd-odd partition function (see (10)) is proportional to  $\left| \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (0, \tau) \right|^2$  and therefore vanishes (see (106)) and  $h(\tau, \bar{\tau})$  is singular because of (165). Suitable regularization gives

$$\left| \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (0, \tau) \right|^2 h(\tau, \bar{\tau}) = C \cdot \left| \partial_v \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (v, \tau) \Big|_{v=0} \right|^2 = C \cdot 4\pi^2 |\eta(\tau)|^6, \quad (46)$$

where  $C$  is a real constant. With (22) we finally arrive at

$$\delta_{\mathcal{T}} = -\pi^2 (\log 3) g_s^2 B_2 C C_m. \quad (47)$$

One the other hand we have (7), i.e.

$$\delta_{\mathcal{T}} = \frac{B_2}{6\pi}. \quad (48)$$

This fixes the matching coefficient

$$C_m = -\frac{1}{6\pi^3 (\log 3) g_s^2 C}. \quad (49)$$

### 3 Non-compact orientifolds with $\Omega J$ projection

In this section we consider D-branes in orientifold models because they are (as far as we know it today) among the best possibilities to get a setup in superstring theory that comes close to the standard model. We will work in the non-compact case because as the matter fields, gauge fields and gravity are localized on the D-branes we will not need to compactify. This will also have the advantage that we will have more models at our disposal. The  $\mathbb{Z}_N$  orbifold action e.g. will no longer have to act crystallographically and the number of D-branes will not be fixed. We will consider a non-standard orientifold projection in order to have D3-branes. For simplicity we assume  $N$  to be odd so that we only have D3-branes and we will assume that the D3-branes are coincident and on top of  $O3_+$ -planes. For the discussed orientifolds of  $\mathbb{Z}_N$  orbifolds we have  $\mathcal{N} = 1$  supersymmetry in  $d = 4$ . After we review the partition functions for annulus, Moebius strip and Klein bottle and find the tadpole conditions in section 3.1 we derive their contributions to the one-loop renormalization of the Planck mass in section 3.2.

#### 3.1 Tadpole conditions

Let  $\Omega$  be the world sheet parity transformation and  $J$  act on the transverse complex bosons  $Z^i = X^{2i+2} + iX^{2i+3}$ ,  $i = 1, 2, 3$  as

$$J Z^i = -Z^i. \quad (50)$$

We consider the  $\Omega J$  orientifold of the non-compact  $\mathbb{Z}_N$  orbifold of type IIB superstring theory and we assume  $N$  to be odd. Therefore we only have D3-branes (for  $N$  even we would also have D7-branes). This model has been presented in [20] (see [21] for a review). The one-loop amplitudes are the torus  $\mathcal{T}$ , the annulus  $\mathcal{A}$ , the Moebius strip  $\mathcal{M}$  and the Klein bottle  $\mathcal{K}$ . The torus contribution to the one-loop renormalization of the Planck mass is one half of the corresponding orbifold result. The  $\mathcal{A}$ ,  $\mathcal{M}$  and  $\mathcal{K}$  partition functions for the standard  $\Omega$  orientifolds (that has only D9 branes if  $N$  is odd) can e.g. be found in [25] (see also [26] and [27]) and we use the same convention as in [25] that we suppress the winding and momentum sums. The annulus, Moebius and Klein bottle amplitudes for the non-compact  $\Omega J$  orientifolds have been presented in [22]. Let us define  $q = e^{2\pi i\tau}$ . For the annulus  $\tau = \frac{1}{2}i\tau_2$ . The annulus partition function is given by

$$Z_{\mathcal{A}} = \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \text{Tr} \left[ (1 + (-1)^F) \theta^k q^{L_0} \right], \quad (51)$$

and we find

$$\begin{aligned}
Z_{\mathcal{A}} &= \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \sum_{\alpha, \beta=0,1} (-1)^{\alpha+\beta+\alpha\beta} \frac{\theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 \end{smallmatrix} \right]}{\eta^3} \\
&\quad \times \prod_{i=1}^3 |2 \sin(\pi k v_i)| \frac{\theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 + k v_i \end{smallmatrix} \right]}{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + k v_i \end{smallmatrix} \right]} (\text{Tr } \gamma_{k,3})^2 \\
&= \frac{(1-1)}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \frac{\theta \left[ \begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] (0, \frac{1}{2} i \tau_2)}{\eta (\frac{1}{2} i \tau_2)^3} \\
&\quad \times \prod_{i=1}^3 |2 \sin(\pi k v_i)| \frac{\theta \left[ \begin{smallmatrix} 0 \\ 1/2 + k v_i \end{smallmatrix} \right] (0, \frac{1}{2} i \tau_2)}{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + k v_i \end{smallmatrix} \right] (0, \frac{1}{2} i \tau_2)} (\text{Tr } \gamma_{k,3})^2.
\end{aligned} \tag{52}$$

For the Moebius amplitude we have  $\tau = \frac{1}{2} + \frac{1}{2} i \tau_2$ . The Moebius partition function is given by

$$Z_{\mathcal{M}} = \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \text{Tr} \left[ (1 + (-1)^F) \Omega J \theta^k q^{L_0} \right]. \tag{54}$$

If we let everything depend on

$$q_{\text{new}} = q_{\text{old}}^2 = e^{4i\pi\tau} = e^{-2\pi\tau_2}, \tag{55}$$

then we find

$$\begin{aligned}
Z_{\mathcal{M}} &= \frac{(1-1)}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \frac{\theta \left[ \begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] (0, i \tau_2) \theta \left[ \begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] (0, i \tau_2)}{\eta (i \tau_2)^3 \theta \left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (0, i \tau_2)} \\
&\quad \times \prod_{i=1}^3 s_i (-2 \sin(\pi k v_i)) \frac{\theta \left[ \begin{smallmatrix} 1/2 \\ k v_i \end{smallmatrix} \right] (0, i \tau_2) \theta \left[ \begin{smallmatrix} 0 \\ 1/2 + k v_i \end{smallmatrix} \right] (0, i \tau_2)}{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + k v_i \end{smallmatrix} \right] (0, i \tau_2) \theta \left[ \begin{smallmatrix} 0 \\ k v_i \end{smallmatrix} \right] (0, i \tau_2)} \text{Tr } \gamma_{\Omega_k,3}^{-1} \gamma_{\Omega_k,3}^T,
\end{aligned} \tag{56}$$

where  $s_i = \text{sign}(\sin(2\pi k v_i))$ . For the Klein bottle we have  $\tau = 2i\tau_2$ . The Klein bottle partition function is given by

$$Z_{\mathcal{K}} = \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{n,k=0}^{N-1} \text{Tr} \left[ (1 + (-1)^F) \Omega J \theta^k q^{L_0(\theta^n)} \bar{q}^{\bar{L}_0(\theta^n)} \right]. \tag{57}$$

$\Omega$  exchanges  $\theta^n$  with  $\theta^{N-n}$ . As we have chosen  $N$  to be odd only  $n = 0$  does survive in the trace. We arrive at

$$Z_K = \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \sum_{\alpha, \beta=0,1} (-1)^{\alpha+\beta+\alpha\beta} \frac{\theta \left[ \frac{\alpha/2}{\beta/2} \right]}{\eta^3} \times \prod_{i=1}^3 \frac{|2 \sin(2\pi k v_i)|}{4(\sin(\pi(k v_i + \frac{1}{2})))^2} \frac{\theta \left[ \frac{\alpha/2}{\beta/2 + 2k v_i} \right]}{\theta \left[ \frac{1/2}{1/2 + 2k v_i} \right]} \quad (58)$$

$$= \frac{(1-1)}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \frac{\theta \left[ \frac{0}{1/2} \right] (0, 2i\tau_2)}{\eta(2i\tau_2)^3} \times \prod_{i=1}^3 \frac{|2 \sin(2\pi k v_i)|}{4(\sin(\pi(k v_i + \frac{1}{2})))^2} \frac{\theta \left[ \frac{0}{1/2 + 2k v_i} \right] (0, 2i\tau_2)}{\theta \left[ \frac{1/2}{1/2 + 2k v_i} \right] (0, 2i\tau_2)}. \quad (59)$$

We show in appendix D that this leads to the tadpole conditions

$$0 = \frac{1}{4} \prod_{i=1}^3 |2 \sin(\pi k v_i)| (\text{Tr } \gamma_{k,3})^2 + 2 \prod_{i=1}^3 s_i (-2 \sin(\pi k v_i)) \text{Tr } (\gamma_{\Omega_k,3}^{-1} \gamma_{\Omega_k,3}^T) + 4 \prod_{i=1}^3 \frac{|2 \sin(2\pi k v_i)|}{4(\sin(\pi(k v_i + \frac{1}{2})))^2}. \quad (60)$$

that are equivalent to

$$0 = \left( \text{Tr } \gamma_{2k,3} \mp 4 \prod_{i=1}^3 \frac{1}{2 \cos(\pi k v_i)} \right)^2. \quad (61)$$

As we are considering the non-compact case we have no untwisted tadpole cancellation condition and the number of D3-branes that we call  $n_3$  is arbitrary. But we still have to impose the twisted tadpole cancellation conditions ( $k = 1, \dots, N-1$ ). For  $\mathbb{Z}_N$  orientifolds we have  $\gamma_{k,3} = \gamma_{1,3}^k$ ,  $k = 1, \dots, N-1$ , and  $\gamma_{1,3}^N = \mathbf{1}$ . Remember that  $\sum_{k=0}^{N-1} e^{2i\pi k/N} = 0$ .

### 3.2 Contribution of $\mathcal{A}$ , $\mathcal{M}$ and $\mathcal{K}$ to the renormalization of the Planck mass

We generalize the results of [23] and [19] that considered compactifications on K3 (and therefore of the  $\mathbb{Z}_2$  orientifold) to general non-compact  $\mathbb{Z}_N$  orientifolds with  $N$  odd.

For  $\mathcal{K}, \mathcal{A}, \mathcal{M}$  there is only one spin structure. The even spin structure two graviton amplitude is given by (see e.g. [15])

$$A^{(2)} = \sum_{(\alpha, \beta)=0,1}^{even} \int_0^\infty \frac{d\tau_2}{\tau_2^2} (-)^{\alpha+\beta+\alpha\beta} Z(\tau, \bar{\tau}, (\alpha, \beta)) \times \int d^2 z_1 \int d^2 z_2 \langle V^{(0,0)}(z_1, \bar{z}_1) V^{(0,0)}(z_2, \bar{z}_2) \rangle_{(\alpha, \beta)}. \quad (62)$$

The piece in second order in momentum will give us the one-loop renormalization of the Planck mass. Neglecting terms proportional to  $p_1^2, p_2^2, p_1 \cdot p_2, p_{1\mu} \varepsilon_1^{\mu\nu}, p_{2\rho} \varepsilon_2^{\rho\sigma}, \eta_{\mu\nu} \varepsilon_1^{\mu\nu}, \eta_{\rho\sigma} \varepsilon_2^{\rho\sigma}$  we find

$$\begin{aligned} A^{(2)}(\tau, \bar{\tau}, (\alpha, \beta)) \Big|_{O(p^2)} &= \int d^2 z_1 \int d^2 z_2 \langle V^{(0,0)}(z_1, \bar{z}_1) V^{(0,0)}(z_2, \bar{z}_2) \rangle_{(\alpha, \beta)} \Big|_{O(p^2)} \\ &= \sum_{\sigma=\mathcal{K}, \mathcal{A}, \mathcal{M}} g_s^2 \int d^2 z_1 \int d^2 z_2 \varepsilon_{\mu\nu}^1 \varepsilon_{\rho\sigma}^2 p_2^\nu p_1^\sigma \eta^{\mu\rho} \left[ \langle \partial X \partial X \rangle \langle \tilde{\psi} \tilde{\psi} \rangle_{(\alpha, \beta)}^2 \right. \\ &\quad \left. - \langle \bar{\partial} X \partial X \rangle \langle \psi \tilde{\psi} \rangle_{(\alpha, \beta)}^2 + \langle \bar{\partial} X \bar{\partial} X \rangle \langle \psi \psi \rangle_{(\alpha, \beta)}^2 - \langle \partial X \bar{\partial} X \rangle \langle \tilde{\psi} \psi \rangle_{(\alpha, \beta)}^2 \right]. \quad (63) \end{aligned}$$

From now on we again measure everything in string units. For  $\mathcal{K}, \mathcal{A}, \mathcal{M}$  we have to act with the following involutions on the covering tori

$$I_{\mathcal{A}} = I_{\mathcal{M}} = 1 - \bar{z}, \quad I_{\mathcal{K}} = 1 - \bar{z} + \frac{\tau}{2}. \quad (64)$$

If on the covering torus we have  $\langle \psi(z) \psi(w) \rangle_{\mathcal{T}, (\alpha, \beta)} = P_F((\alpha, \beta); z, w)$  then by the method of images (see appendix of [23])

$$\langle \psi(z) \psi(w) \rangle_{\sigma, (\alpha, \beta)} = P_F((\alpha, \beta); z, w) \quad (65)$$

$$\langle \psi(z) \tilde{\psi}(\bar{w}) \rangle_{\sigma, (\alpha, \beta)} = P_F((\alpha, \beta); z, I_\sigma(w)) \quad (66)$$

$$\langle \tilde{\psi}(\bar{z}) \tilde{\psi}(\bar{w}) \rangle_{\sigma, (\alpha, \beta)} = \bar{P}_F((\bar{\alpha}, \bar{\beta}); \bar{z}, \bar{w}). \quad (67)$$

On the other hand

$$(\langle \psi(z) \psi(0) \rangle_{\mathcal{T}, (\alpha, \beta)})^2 = -\partial_z^2 \log \theta \left[ \frac{1/2}{1/2} \right] (z, \tau) + 4\pi i \partial_\tau \log \theta \left[ \frac{\alpha/2}{\beta/2} \right] (0, \tau) \quad (68)$$

i.e. it can be written as a sum of a term that is independent of the spin structure (but dependent on the position  $z$  on the world-sheet) and therefore vanishes when summed over the spin structure (as the partition function vanishes due to supersymmetry) and a term independent of the position  $z$  on the world-sheet (but dependent on the spin structure) that can be taken outside the world-sheet integral. The surviving piece will be the same in  $\langle \tilde{\psi} \tilde{\psi} \rangle_{(\alpha, \beta)}^2, \langle \psi \tilde{\psi} \rangle_{(\alpha, \beta)}^2, \langle \psi \psi \rangle_{(\alpha, \beta)}^2, \langle \tilde{\psi} \psi \rangle_{(\alpha, \beta)}^2$  so we replace it by  $\langle \psi \psi \rangle_{(\alpha, \beta)}^2$  everywhere.

The remaining integral over the bosonic correlators gives (see again [23])

$$\int d^2 z_1 \int d^2 z_2 \left[ \langle \partial X \partial X \rangle - \langle \bar{\partial} X \partial X \rangle + \langle \bar{\partial} X \bar{\partial} X \rangle - \langle \partial X \bar{\partial} X \rangle \right] = \begin{cases} \pi \tau_2 / 8 & \text{for } \sigma = \mathcal{A}, \mathcal{M} \\ \pi \tau_2 / 2 & \text{for } \sigma = \mathcal{K} \end{cases}. \quad (69)$$

Using (33) and (105) we find for the annulus

$$\delta_{\mathcal{A}} = -g_s^2 C_m \sum_{(\alpha, \beta)=0,1}^{\text{even}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} (-)^{\alpha+\beta+\alpha\beta} Z_{\mathcal{A}}(\tau, \bar{\tau}, (\alpha, \beta)) \frac{\partial_v^2 \theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 \end{smallmatrix} \right] (v, \frac{1}{2}i\tau_2) \Big|_{v=0}}{\theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 \end{smallmatrix} \right] (0, \frac{1}{2}i\tau_2)} \frac{\pi \tau_2}{8}, \quad (70)$$

for the Moebius strip

$$\delta_{\mathcal{M}} = -g_s^2 C_m \sum_{(\alpha, \beta)=0,1}^{\text{even}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} (-)^{\alpha+\beta+\alpha\beta} Z_{\mathcal{M}}(\tau, \bar{\tau}, (\alpha, \beta)) \frac{\partial_v^2 \theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 \end{smallmatrix} \right] (v, \frac{1}{2}i\tau_2 + \frac{1}{2}) \Big|_{v=0}}{\theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 \end{smallmatrix} \right] (0, \frac{1}{2}i\tau_2 + \frac{1}{2})} \frac{\pi \tau_2}{8} \quad (71)$$

and for the Klein bottle

$$\delta_{\mathcal{K}} = -g_s^2 C_m \sum_{(\alpha, \beta)=0,1}^{\text{even}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} (-)^{\alpha+\beta+\alpha\beta} Z_{\mathcal{K}}(\tau, \bar{\tau}, (\alpha, \beta)) \frac{\partial_v^2 \theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 \end{smallmatrix} \right] (v, 2i\tau_2) \Big|_{v=0}}{\theta \left[ \begin{smallmatrix} \alpha/2 \\ \beta/2 \end{smallmatrix} \right] (0, 2i\tau_2)} \frac{\pi \tau_2}{2}. \quad (72)$$

Using (106) to (108), the Riemann identity (109) and the partition functions (52), (56) and (58) we get for the annulus

$$\begin{aligned} \delta_{\mathcal{A}} = & -g_s^2 C_m \frac{1}{2N} \partial_v^2 \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=1}^{N-1} \frac{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (\frac{v}{2}, \frac{1}{2}i\tau_2)}{\eta(\frac{1}{2}i\tau_2)^3} \\ & \times \left[ \prod_{i=1}^3 |2 \sin(\pi k v_i)| \frac{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + k v_i \end{smallmatrix} \right] (\frac{v}{2}, \frac{1}{2}i\tau_2)}{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + k v_i \end{smallmatrix} \right] (0, \frac{1}{2}i\tau_2)} \right] (\text{Tr } \gamma_{k,3})^2 \frac{\pi \tau_2}{8} \Big|_{v=0} \end{aligned} \quad (73)$$

for the Moebius strip

$$\delta_{\mathcal{M}} = -g_s^2 C_m \frac{1}{2N} \partial_v^2 \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=1}^{N-1} \frac{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (\frac{v}{2}, i\tau_2) \theta \left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (\frac{v}{2}, i\tau_2)}{\eta(i\tau_2)^3 \theta \left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (0, i\tau_2)}$$

$$\begin{aligned}
& \times \left[ \prod_{i=1}^3 s_i (-2 \sin(\pi k v_i)) \frac{\theta \begin{bmatrix} 1/2 \\ 1/2 + k v_i \end{bmatrix} \left( \frac{v}{2}, i \tau_2 \right) \theta \begin{bmatrix} 0 \\ k v_i \end{bmatrix} \left( \frac{v}{2}, i \tau_2 \right)}{\theta \begin{bmatrix} 1/2 \\ 1/2 + k v_i \end{bmatrix} (0, i \tau_2) \theta \begin{bmatrix} 0 \\ k v_i \end{bmatrix} (0, i \tau_2)} \right] \\
& \times \left. \text{Tr } \gamma_{\Omega_k, 3}^{-1} \gamma_{\Omega_k, 3}^T \frac{\pi \tau_2}{8} \right|_{v=0}
\end{aligned} \tag{74}$$

and for the Klein bottle

$$\begin{aligned}
\delta_{\mathcal{K}} &= -g_s^2 C_m \frac{1}{2N} \partial_v^2 \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=1}^{N-1} \frac{\theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \left( \frac{v}{2}, 2i\tau_2 \right)}{\eta(2i\tau_2)^3} \\
& \times \left[ \prod_{i=1}^3 \frac{|2 \sin(2\pi k v_i)|}{4(\sin(\pi(k v_i + \frac{1}{2})))^2} \frac{\theta \begin{bmatrix} 1/2 \\ 1/2 + 2k v_i \end{bmatrix} \left( \frac{v}{2}, 2i\tau_2 \right)}{\theta \begin{bmatrix} 1/2 \\ 1/2 + 2k v_i \end{bmatrix} (0, 2i\tau_2)} \right] \frac{\pi \tau_2}{2} \Bigg|_{v=0}.
\end{aligned} \tag{75}$$

We go to the transverse channel. For the annulus we have the transformations  $t = \frac{1}{2}\tau_2, l = \frac{1}{t}$

$$\begin{aligned}
\delta_{\mathcal{A}} &= -g_s^2 C_m \frac{1}{2^2} \frac{1}{2N} \partial_v^2 \int_0^\infty dl \sum_{k=1}^{N-1} \frac{(-i) e^{-\frac{\pi l v^2}{4}} \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \left( \frac{i v l}{2}, i l \right)}{\eta(i l)^3} \\
& \times \left[ \prod_{i=1}^3 |2 \sin(\pi k v_i)| \frac{e^{-\frac{\pi l v^2}{4}} \theta \begin{bmatrix} 1/2 + k v_i \\ 1/2 \end{bmatrix} \left( \frac{i v l}{2}, i l \right)}{\theta \begin{bmatrix} 1/2 + k v_i \\ 1/2 \end{bmatrix} (0, i l)} \right] (\text{Tr } \gamma_{k, 3})^2 \frac{\pi}{4l} \Bigg|_{v=0}.
\end{aligned} \tag{76}$$

For the Moebius strip we have the transformations  $t = \frac{1}{\tau_2}, l = \frac{t}{2}$

$$\begin{aligned}
\delta_{\mathcal{M}} &= -g_s^2 C_m 2 \frac{1}{2N} \partial_v^2 \int_0^\infty dl \sum_{k=1}^{N-1} \frac{(-i) e^{-\frac{\pi l v^2}{2}} \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (i v l, 2i l) e^{-\frac{\pi l v^2}{2}} \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (i v l, 2i l)}{\eta(2i l)^3 \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2i l)} \\
& \times \left[ \prod_{i=1}^3 s_i (-2 \sin(\pi k v_i)) \frac{e^{-\frac{\pi l v^2}{2}} \theta \begin{bmatrix} 1/2 + k v_i \\ 1/2 \end{bmatrix} (i v l, 2i l) e^{-\frac{\pi l v^2}{2}} \theta \begin{bmatrix} k v_i \\ 0 \end{bmatrix} (i v l, 2i l)}{\theta \begin{bmatrix} 1/2 + k v_i \\ 1/2 \end{bmatrix} (0, 2i l) \theta \begin{bmatrix} k v_i \\ 0 \end{bmatrix} (0, 2i l)} \right] \\
& \times \left. \text{Tr } \gamma_{\Omega_k, 3}^{-1} \gamma_{\Omega_k, 3}^T \frac{\pi}{16l} \right|_{v=0}.
\end{aligned} \tag{77}$$

For the Klein bottle we have the transformations  $t = 2\tau_2, l = \frac{1}{t}$

$$\begin{aligned} \delta_{\mathcal{K}} = & -g_s^2 C_m 2^2 \frac{1}{2N} \partial_v^2 \int_0^\infty dl \sum_{k=1}^{N-1} \frac{(-i) e^{-\frac{\pi l v^2}{4}} \theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{ivl}{2}, il \right)}{\eta(il)^3} \\ & \times \left[ \prod_{i=1}^3 \frac{|2 \sin(2\pi k v_i)|}{4(\sin(\pi(k v_i + \frac{1}{2})))^2} \frac{e^{-\frac{\pi l v^2}{4}} \theta \left[ \begin{smallmatrix} 1/2 + 2k v_i \\ 1/2 \end{smallmatrix} \right] \left( \frac{ivl}{2}, il \right)}{\theta \left[ \begin{smallmatrix} 1/2 + 2k v_i \\ 1/2 \end{smallmatrix} \right] (0, il)} \right] \frac{\pi}{4l} \Big|_{v=0}. \end{aligned} \quad (78)$$

Note that it is important here to take the derivatives with respect to  $v$  only after one goes to the transverse channel. Using (98) to (108) we arrive at

$$\begin{aligned} \delta_{\mathcal{A}} = & -2\pi i g_s^2 C_m \frac{1}{2^2} \frac{1}{2N} \int_0^\infty dl \sum_{k=1}^{N-1} \left( \frac{il}{2} \right)^2 \left[ \prod_{j=1}^3 |2 \sin(\pi k v_j)| \right] \\ & \times \sum_{i=1}^3 \left( i\pi + 2i\pi k v_i + f \left( \frac{1}{2} + k v_i, \frac{1}{2}, il \right) \right) (\text{Tr } \gamma_{k,3})^2 \frac{\pi}{4l} \end{aligned} \quad (79)$$

$$\begin{aligned} \delta_{\mathcal{M}} = & -2\pi i g_s^2 C_m 2 \frac{1}{2N} \int_0^\infty dl \sum_{k=1}^{N-1} (il)^2 \left[ \prod_{j=1}^3 s_j(-2 \sin(\pi k v_j)) \right] \\ & \times \left[ f(0, 0, 2il) + \sum_{i=1}^3 \left( i\pi + 2i\pi k v_i + f \left( \frac{1}{2} + k v_i, \frac{1}{2}, 2il \right) \right) \right. \\ & \left. + \sum_{i=1}^3 (2i\pi k v_i + f(k v_i, 0, 2il)) \right] \text{Tr } \gamma_{\Omega_k,3}^{-1} \gamma_{\Omega_k,3}^T \frac{\pi}{16l} \end{aligned} \quad (80)$$

$$\begin{aligned} \delta_{\mathcal{K}} = & -2\pi i g_s^2 C_m 2^2 \frac{1}{2N} \int_0^\infty dl \sum_{k=1}^{N-1} \left( \frac{il}{2} \right)^2 \left[ \prod_{j=1}^3 \frac{|2 \sin(2\pi k v_j)|}{4(\sin(\pi(k v_j + \frac{1}{2})))^2} \right] \\ & \times \sum_{i=1}^3 \left( i\pi + 4i\pi k v_i + f \left( \frac{1}{2} + 2k v_i, \frac{1}{2}, il \right) \right) \frac{\pi}{4l}. \end{aligned} \quad (81)$$

The tadpole cancellation condition (60) guarantees that the contribution of terms in the sum proportional to a constant (see the  $i\pi$ ) in  $\delta = \delta_{\mathcal{A}} + \delta_{\mathcal{M}} + \delta_{\mathcal{K}}$  vanishes. On the other hand  $\sum_{i=1}^3 v_i = 0$  because we consider supersymmetric models. All remaining terms in  $\delta$  are proportional to some  $f(a, b, il)$ . These functions (for  $a \in (-1/2, 1/2)$ ) fall off rapidly as  $l \rightarrow \infty$ . Therefore  $\delta$  is



free of ultraviolet divergences due to tadpole cancellation. We are left with

$$\begin{aligned}
\delta &= \frac{i\pi^2}{16N} g_s^2 C_m \int_0^\infty dl l \sum_{k=1}^{N-1} \left\{ \frac{1}{2^2} \left[ \prod_{j=1}^3 |2 \sin(\pi k v_j)| \right] (\text{Tr } \gamma_{k,3})^2 \right. \\
&\quad \times \sum_{i=1}^3 f\left(\frac{1}{2} + k v_i, \frac{1}{2}, il\right) + 2 \left[ \prod_{j=1}^3 s_j(-2 \sin(\pi k v_j)) \right] \left[ f(0, 0, 2il) \right. \\
&\quad \left. + \sum_{i=1}^3 f\left(\frac{1}{2} + k v_i, \frac{1}{2}, 2il\right) + \sum_{i=1}^3 f(k v_i, 0, 2il) \right] \text{Tr } \gamma_{\Omega_k,3}^{-1} \gamma_{\Omega_k,3}^T \\
&\quad \left. + 2^2 \left[ \prod_{j=1}^3 \frac{|2 \sin(2\pi k v_j)|}{4(\sin(\pi(k v_j + \frac{1}{2})))^2} \right] \sum_{i=1}^3 f\left(\frac{1}{2} + 2k v_i, \frac{1}{2}, il\right) \right\}. \quad (82)
\end{aligned}$$

Explicitly using the tadpole cancellation conditions (61) we arrive at

$$\begin{aligned}
\delta &= \frac{i\pi^2}{4N} g_s^2 C_m \int_0^\infty dl l \sum_{k=1}^{N-1} \left[ \prod_{j=1}^3 |\tan(\pi k v_j)| \right] \left\{ \sum_{i=1}^3 f\left(\frac{1}{2} + k v_i, \frac{1}{2}, il\right) \right. \\
&\quad - 2 \left[ \sum_{i=1}^3 f\left(\frac{1}{2} + k v_i, \frac{1}{2}, 2il\right) + \sum_{i=1}^3 f(k v_i, 0, 2il) \right] \\
&\quad \left. + \sum_{i=1}^3 f\left(\frac{1}{2} + 2k v_i, \frac{1}{2}, il\right) \right\}. \quad (83)
\end{aligned}$$

Though the integral is free of ultraviolet divergences we may still have some infrared divergences that can only be handled by considering the Wilsonian couplings.  $C_m$  is given by (49). For the non-compact  $\mathbb{Z}_N$  orientifolds with odd  $N$  we can estimate from (83) the large  $N$  behavior

$$\delta = \delta_{\mathcal{A}} + \delta_{\mathcal{M}} + \delta_{\mathcal{K}} \xrightarrow{N \rightarrow \infty} O(1) \quad (84)$$

that is subleading as compared to the torus contribution. By using  $\sum_{i=1}^3 v_i = 0$  and the properties (100) to (102) one can actually check that the contribution from the sector  $(N - k)$  cancels the contribution from the sector  $k$ , i.e.

$$\delta = \delta_{\mathcal{A}} + \delta_{\mathcal{M}} + \delta_{\mathcal{K}} = 0. \quad (85)$$

The one-loop renormalization of the four-dimensional Planck mass comes only from the torus that is given by one half of the orbifold result (27) and is  $O(N)$ .

## 4 Conclusions

We have considered D-branes in orientifold models because they are (as far as we know it today) among the best possibilities to get a setup in superstring theory that comes close to the standard model. We focused on the non-compact case because in these models the matter fields, gauge fields and gravity are localized on the D-branes and we do not need to compactify. The issue was to show that the one-loop correction of the Planck mass can be arbitrary large in string units. It is therefore possible to accommodate the measured four-dimensional Planck mass as a one-loop effect and to have a string scale far below the Planck scale.

To be more precise we have constructed non-compact orientifolds of  $\mathbb{Z}_N$  orbifolds of type IIB with induced gravity on coincident D3-branes that are on top of  $O3_+$ -planes. That we consider orientifolds of  $\mathbb{Z}_N$  orbifolds is because they have localized twisted sectors and therefore localized gravity. As we consider the non-compact case the orbifold need not act crystallographically. That we assumed  $N$  to be odd and the D3-branes to be coincident and on top of  $O3_+$ -planes was just for simplicity.

We have shown for the  $\mathbb{Z}_N$  orientifolds with odd  $N$  that the contribution to the one-loop renormalization of the four-dimensional Planck mass comes only from the torus and is  $O(N)$  as the contributions from annulus, Moebius strip and Klein bottle cancel. The idea that four-dimensional gravity may be induced by quantum corrections at the one-loop level can therefore be realized by considering sufficiently large  $N$ .

Obviously the models presented in this paper are only toy models of orientifold realizations of the standard model and there is plenty of room for generalization. The aim will be to construct more realistic brane induced gravity models that come closer to (supersymmetric generalizations) of the standard model by considering e.g. more general D-brane configurations, Scherk-Schwarz directions or Wilson lines. One will have to check the higher loop corrections to the Planck mass and also the renormalization of higher derivative terms as e.g. the  $R^2$ -terms.

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## A Theta functions

We use the definitions of [28]

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{n=-\infty}^{\infty} \exp \left[ i\pi(n+a)^2\tau + 2\pi i(n+a)(z+b) \right] \quad (86)$$

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (87)$$

This gives

$$\theta \begin{bmatrix} -a \\ -b \end{bmatrix} (z, \tau) = \theta \begin{bmatrix} a \\ b \end{bmatrix} (-z, \tau) \quad (88)$$

$$\theta \begin{bmatrix} a+1 \\ b \end{bmatrix} (z, \tau) = \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) \quad (89)$$

$$\theta \begin{bmatrix} a \\ b+1 \end{bmatrix} (z, \tau) = e^{2i\pi a} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau). \quad (90)$$

We have the modular transformations

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau + 1) = e^{-i\pi(a^2+a)} \theta \begin{bmatrix} a \\ \frac{1}{2} + a + b \end{bmatrix} (z, \tau) \quad (91)$$

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} \left( \frac{z}{\tau}, -\frac{1}{\tau} \right) = (-i\tau)^{1/2} \exp \left[ 2\pi iab + \frac{i\pi z^2}{\tau} \right] \theta \begin{bmatrix} b \\ -a \end{bmatrix} (z, \tau) \quad (92)$$

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau) \quad (93)$$

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2} \eta(\tau), \quad (94)$$

where the second property is shown using Poisson resummation

$$\sum_{n=-\infty}^{\infty} \exp \left[ -\pi an^2 + 2\pi ibn \right] = a^{-1/2} \sum_{m=-\infty}^{\infty} \exp \left[ -\frac{\pi(m-b)^2}{a} \right]. \quad (95)$$

The theta-functions have the product representation

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (v, \tau) = e^{2i\pi ab} q^{\frac{a^2}{2}} z^a \prod_{m=1}^{\infty} (1 - q^m) \left( 1 + e^{2\pi ib} z q^{m-\frac{1}{2}+a} \right) \left( 1 + e^{-2\pi ib} z^{-1} q^{m-\frac{1}{2}-a} \right), \quad (96)$$

where  $q = e^{2i\pi\tau}$ ,  $z = e^{2i\pi v}$ . In particular we have

$$\frac{\theta \begin{bmatrix} a \\ b \end{bmatrix} (0, \tau)}{\eta(\tau)} = e^{2i\pi ab} q^{\frac{a^2}{2} - \frac{1}{24}} \prod_{m=1}^{\infty} \left[ 1 + e^{-2\pi ib} q^{(m-\frac{1}{2}-a)} \right] \left[ 1 + e^{2\pi ib} q^{(m-\frac{1}{2}+a)} \right]. \quad (97)$$

For the derivatives we get

$$\left. \frac{\partial_v \theta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] (v, \tau)}{\theta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] (v, \tau)} \right|_{v=0} = 2i\pi a + f(a, b; \tau), \quad (98)$$

where

$$f(a, b; \tau) = 2\pi i \sum_{m=1}^{\infty} \left[ \frac{e^{2i\pi b} q^{m-\frac{1}{2}+a}}{1 + e^{2i\pi b} q^{m-\frac{1}{2}+a}} - \frac{e^{-2i\pi b} q^{m-\frac{1}{2}-a}}{1 + e^{-2i\pi b} q^{m-\frac{1}{2}-a}} \right]. \quad (99)$$

From the behaviour of the theta functions follows

$$f(-a, -b; \tau) = -f(a, b; \tau) \quad (100)$$

$$f(a+1, b; \tau) = -2i\pi + f(a, b; \tau) \quad (101)$$

$$f(a, b+1; \tau) = f(a, b; \tau) \quad (102)$$

$$f(a, b; \tau+1) = f(a, 1/2 + a + b; \tau) \quad (103)$$

$$f(a, b; -1/\tau) = 2i\pi(b-a) + f(b, -a; \tau). \quad (104)$$

The theta functions are solutions of the heat equation

$$\frac{\partial^2}{\partial z^2} \theta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z, \tau) = 4\pi i \frac{\partial}{\partial \tau} \theta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z, \tau) \quad (105)$$

moreover

$$\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (0, \tau) = 0 \quad (106)$$

$$\left. \partial_v \theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (v, \tau) \right|_{v=0} = -2\pi\eta(\tau)^3 \quad (107)$$

$$\left. \partial_v^2 \theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (v, \tau) \right|_{v=0} = 0. \quad (108)$$

For  $\sum_{i=1}^4 h_i = \sum_{i=1}^4 g_i = 0$  we have the Riemann identity

$$\frac{1}{2} \sum_{\alpha, \beta=0}^1 (-)^{\alpha+\beta+\alpha\beta} \prod_{i=1}^4 \theta \left[ \begin{smallmatrix} \alpha/2 + h_i \\ \beta/2 + g_i \end{smallmatrix} \right] (v_i) = - \prod_{i=1}^4 \theta \left[ \begin{smallmatrix} 1/2 - h_i \\ 1/2 - g_i \end{smallmatrix} \right] (v'_i) \quad (109)$$

$$v'_1 = \frac{1}{2}(-v_1 + v_2 + v_3 + v_4), \quad v'_2 = \frac{1}{2}(v_1 - v_2 + v_3 + v_4) \quad (110)$$

$$v'_3 = \frac{1}{2}(v_1 + v_2 - v_3 + v_4), \quad v'_4 = \frac{1}{2}(v_1 + v_2 + v_3 - v_4). \quad (111)$$

## B The normalization of the partition function

### B.1 The example of the $\mathbb{Z}_3$ orbifold

Let us first consider the compact case.

#### B.1.1 The massless spectrum

Let  $T^6 = T^2 \times T^2 \times T^2$  and let us define  $\alpha = e^{\frac{2\pi i}{3}}$ . The  $\mathbb{Z}_3$  orbifold acts on the tori as  $Z^i \rightarrow e^{2\pi i v_i} Z^i$ , where  $v = (v_1, v_2, v_3) = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ , i.e. we have the reflection

$$r : Z^1 \rightarrow \alpha Z^1, Z^2 \rightarrow \alpha Z^2, Z^3 \rightarrow \alpha^{-2} Z^3. \quad (112)$$

For the orbifold to act crystallographically the torus moduli of the three spacetime tori have to be  $\tau_i = \alpha^{1/2} R_i$ . For each torus we have 3 fixed points at  $n\alpha^{1/4} R_i/3$ ,  $n = 0, 1, 2$ , giving a total of  $3^3 = 27$  fixed points.

#### The untwisted massless spectrum

The zero point energy of a complex boson with twist  $\theta$  is

$$f(\theta) = \frac{1 - 3(1 - 2\theta)^2}{24} \quad (113)$$

and the negative of this for a complex fermion. The shift is zero for the 4 complex bosons (the transverse and the three compact) so the zero point energy is in the NS sector

$$4f(0) - 4f(1/2) = -\frac{1}{2} \quad (114)$$

and the first excited states are massless. In the R sector instead we have the zero point energy

$$4f(0) - 4f(0) = 0 \quad (115)$$

and the massless states are the degenerate ground states.

Let us separate the lefthanded part of the massless states according to their eigenvalue under  $\alpha$ :

$$\alpha^0 : \psi_{-1/2}^\mu |0\rangle_{NS}, \left| \frac{1}{2}, \mathbf{1} \right\rangle_R, \left| -\frac{1}{2}, \bar{\mathbf{1}} \right\rangle_R \quad (116)$$

$$\alpha^1 : \psi_{-1/2}^i |0\rangle_{NS}, \left| \frac{1}{2}, \mathbf{3} \right\rangle_R \quad (117)$$

$$\alpha^2 : \psi_{-1/2}^{\bar{i}} |0\rangle_{NS}, \left| -\frac{1}{2}, \bar{\mathbf{3}} \right\rangle_R, \quad (118)$$

where

$$|\mathbf{1}\rangle_R = \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle_R \quad (119)$$

$$|\bar{\mathbf{1}}\rangle_R = \left| -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle_R \quad (120)$$

$$|\mathbf{3}\rangle_R = \left\{ \left| \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle_R, \left| -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle_R, \left| -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\rangle_R \right\} \quad (121)$$

$$|\bar{\mathbf{3}}\rangle_R = \left\{ \left| -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle_R, \left| \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\rangle_R, \left| \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle_R \right\}. \quad (122)$$

The untwisted massless states that are invariant under the orbifold action come from  $\alpha^0\alpha^0$ ,  $\alpha^1\alpha^2$  and  $\alpha^2\alpha^1$ . We find 44 bosonic states and 44 fermionic states that give the following  $\mathcal{N} = 2$  multiplets

$$\begin{aligned} & \left[ \left( -2, -\frac{3^2}{2}, -1 \right) + \left( 1, \frac{3^2}{2}, 2 \right) \right] + \left[ \left( -1, -\frac{1^2}{2}, 0 \right) + \left( 0, \frac{1^2}{2}, 1 \right) \right]^9 \\ & + \left[ \left( -\frac{1}{2}, 0^2, \frac{1}{2} \right) + \left( -\frac{1}{2}, 0^2, \frac{1}{2} \right) \right], \end{aligned} \quad (123)$$

where the superscripts are not powers but give the number of fields of given helicity.

### The twisted massless spectrum

The massless spectrum is 27 copies of the massless spectrum at one of the fixed points. Let us first consider the states twisted by  $r$ . The transverse complex bosons has shift 0 and the three compact complex bosons have shift  $1/3$ . In the NS sector we get using (113) the zero point energy

$$f(0) - f(1/2) + 3f(1/3) - 3f(1/6) = 0 \quad (124)$$

and in the R sector

$$f(0) - f(0) + 3f(1/3) - 3f(1/3) = 0 \quad (125)$$

so in both cases the massless states are ground states. Let us first consider the R case. There are 2 fermion zero modes coming from the transverse complex fermion leading to 2 possible states  $|\pm \frac{1}{2}\rangle_{h,R}$ . The GSO projection then only leaves only one state  $|\frac{1}{2}\rangle_{h,R}$ . In the NS case there is a unique ground state  $|0\rangle_{h,NS}$ . We find two bosonic states and two fermionic states. The states twisted by  $r^2$  give the antiparticles of these. The total massless spectrum from the twisted states is in terms of  $\mathcal{N} = 2$  multiplets

$$\left[ \left( -1, -\frac{1^2}{2}, 0 \right) + \left( 0, \frac{1^2}{2}, 1 \right) \right]^{27}. \quad (126)$$

### B.1.2 The helicity generating partition function

For type IIB on  $M^4 \times T^6/\mathbb{Z}_3$  with  $T^6 = T^2 \times T^2 \times T^2$  we find

$$Z^{(0,0)}(v, \bar{v}) = N_0(N) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_X^2(\tau) \left[ Z_\psi^+(v, \tau) Z_\psi^+(v, \tau)^* \right] \Big|_{h=g=0} \prod_{i=1}^3 Z_i \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) \quad (127)$$

$$Z'(v, \bar{v}) = N_0(N) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_X^2(\tau) \sum_{\substack{h, g=0 \\ (h, g) \neq (0,0)}}^{N-1} Z_\psi^+(v, \tau) Z_\psi^+(v, \tau)^* \prod_{i=1}^3 Z_i \begin{bmatrix} hv_i \\ gv_i \end{bmatrix} (\tau), \quad (128)$$

where

$$Z_X^2(\tau) = \frac{1}{\tau_2} \frac{1}{|\eta(\tau)|^4} \quad (129)$$

$$Z_i \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) = \frac{\Gamma_{2,2}}{|\eta(\tau)|^4} \quad (\Gamma_{2,2} \text{ is the } (2,2) \text{ lattice sum}) \quad (130)$$

$$Z_i \begin{bmatrix} p \\ q \end{bmatrix} (\tau) = -3 \left| \frac{\eta(\tau)}{\theta \begin{bmatrix} 1/2 + p \\ 1/2 + q \end{bmatrix} (0, \tau)} \right|^2 \quad \text{for } (p, q) \neq (0, 0) \quad (131)$$

$$\begin{aligned} Z_\psi^+(v, \tau) &= \frac{\xi(v)}{2} \frac{1}{\eta(\tau)^4} \sum_{\alpha, \beta=0}^1 (-)^{\alpha+\beta+\alpha\beta} \theta \begin{bmatrix} \alpha/2 \\ \beta/2 \end{bmatrix} (v, \tau) \theta \begin{bmatrix} \alpha/2 + h/3 \\ \beta/2 + g/3 \end{bmatrix} (0, \tau) \\ &\quad \times \theta \begin{bmatrix} \alpha/2 + h/3 \\ \beta/2 + g/3 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \alpha/2 - 2h/3 \\ \beta/2 - 2g/3 \end{bmatrix} (0, \tau). \end{aligned} \quad (132)$$

Using the Riemann identity we get

$$Z_\psi^+(v, \tau) = \frac{\xi(v)}{\eta(\tau)^4} \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (v/2, \tau) \left( \theta \begin{bmatrix} 1/2 - h/3 \\ 1/2 - g/3 \end{bmatrix} (v/2, \tau) \right)^2 \theta \begin{bmatrix} 1/2 + 2h/3 \\ 1/2 + 2g/3 \end{bmatrix} (v/2, \tau). \quad (133)$$

We find the contribution to the helicity generating partition function from the massless modes from the limit  $\tau_2 \rightarrow \infty$ . The twisted states come from  $h = 1, 2$  and the untwisted from  $h = 0$ . Using

$$\Gamma_{2,2}|_{\text{massless}} = 1, \quad \xi(v) \xrightarrow{\tau_2 \rightarrow \infty} 1, \quad \bar{\xi}(\bar{v}) \xrightarrow{\tau_2 \rightarrow \infty} 1 \quad (134)$$

$$\frac{1}{|\eta(\tau)|^6} \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (v/2, \tau) \bar{\theta} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\bar{v}/2, \tau) = 4 \sin \frac{\pi v}{2} \sin \frac{\pi \bar{v}}{2} (1 + O(q\bar{q})) \quad (135)$$

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3} = \log 3 \quad (136)$$

and

$$\left| \frac{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/6 \end{smallmatrix} \right] (v/2, \tau)}{\theta \left[ \begin{smallmatrix} 1/2 \\ 1/6 \end{smallmatrix} \right] (0, \tau)} \right| \xrightarrow{\tau_2 \rightarrow \infty} \frac{1}{\sqrt{3}} \left| e^{i\pi v/2} + e^{-i\pi/3} e^{-i\pi v/2} \right| \quad (137)$$

$$\left| \frac{\theta \left[ \begin{smallmatrix} 1/2 \\ 5/6 \end{smallmatrix} \right] (v/2, \tau)}{\theta \left[ \begin{smallmatrix} 1/2 \\ 5/6 \end{smallmatrix} \right] (0, \tau)} \right| \xrightarrow{\tau_2 \rightarrow \infty} \frac{1}{\sqrt{3}} \left| e^{-i\pi/3} e^{i\pi v/2} + e^{-i\pi v/2} \right| \quad (138)$$

$$\left| \frac{\theta \left[ \begin{smallmatrix} 1/6 \\ b \end{smallmatrix} \right] (v/2, \tau)}{\theta \left[ \begin{smallmatrix} 1/6 \\ b \end{smallmatrix} \right] (0, \tau)} \right| \xrightarrow{\tau_2 \rightarrow \infty} \left| e^{i\pi v/6} \right| \quad (139)$$

$$\left| \frac{\theta \left[ \begin{smallmatrix} 5/6 \\ b \end{smallmatrix} \right] (v/2, \tau)}{\theta \left[ \begin{smallmatrix} 5/6 \\ b \end{smallmatrix} \right] (0, \tau)} \right| \xrightarrow{\tau_2 \rightarrow \infty} \left| e^{-i\pi v/6} \right| \quad (140)$$

that follows from the product representation of the theta functions (96) we find the massless contribution of the twisted sector

$$\begin{aligned} Z^T(v, \bar{v})|_{\text{massless}} &= 3N_0 (-3)^3 4 \left| \sin \frac{\pi v}{2} \right|^2 \left| e^{i\pi v/6} \right|^6 \\ &+ 3N_0 (-3)^3 4 \left| \sin \frac{\pi v}{2} \right|^2 \left| e^{-i\pi v/6} \right|^6 \end{aligned} \quad (141)$$

and the massless contribution of the untwisted sector

$$\begin{aligned} Z^U(v, \bar{v})|_{\text{massless}} &= N_0 256 \left| \sin \frac{\pi v}{2} \right|^8 \\ &- N_0 4 \left| \sin \frac{\pi v}{2} \right|^2 \left| e^{i\pi v/2} + e^{-i\pi/3} e^{-i\pi v/2} \right|^6 \\ &- N_0 4 \left| \sin \frac{\pi v}{2} \right|^2 \left| e^{-i\pi/3} e^{i\pi v/2} + e^{-i\pi v/2} \right|^6. \end{aligned} \quad (142)$$

If we write a function  $f(v, \bar{v})$  as

$$f(v, \bar{v}) = \left( \sum_{\lambda_R} \tilde{c}_{\lambda_R} e^{2i\pi v \lambda_R} \right) \left( \sum_{\lambda_L} c_{\lambda_L} e^{-2i\pi \bar{v} \lambda_L} \right) \quad (143)$$

with coefficients  $\tilde{c}_{\lambda_R}, c_{\lambda_L}$  then the contribution to the fixed helicity  $\lambda_{\text{tot}} = \lambda_R + \lambda_L$  is

$$\sum_{\lambda_R} \tilde{c}_{\lambda_R} c_{(\lambda_{\text{tot}} - \lambda_R)} e^{2i\pi v \lambda_R} e^{-2i\pi \bar{v} (\lambda_{\text{tot}} - \lambda_R)} \Big|_{v=\bar{v}=0} = \sum_{\lambda_R} \tilde{c}_{\lambda_R} c_{(\lambda_{\text{tot}} - \lambda_R)}. \quad (144)$$



The helicity content of  $4 \left| \sin \frac{\pi v}{2} \right|^2 \left| e^{i\pi v/6} \right|^6$  is

$\lambda_{\text{tot}}$	0	1/2	1
value	-1	2	-1

The helicity content of  $4 \left| \sin \frac{\pi v}{2} \right|^2 \left| e^{-i\pi v/6} \right|^6$  is

$\lambda_{\text{tot}}$	0	-1/2	-1
value	-1	2	-1

Comparing with the twisted spectrum (126) fixes the normalization to be

$$N_0 = \frac{1}{3}. \quad (145)$$

The helicity content of  $4 \left| \sin \frac{\pi v}{2} \right|^2 \left| e^{i\pi v/2} + e^{-i\pi/3} e^{-i\pi v/2} \right|^6$  and of  $4 \left| \sin \frac{\pi v}{2} \right|^2 \left| e^{-i\pi/3} e^{i\pi v/2} + e^{-i\pi v/2} \right|^6$  both give

$\lambda_{\text{tot}}$	0	$\pm 1/2$	$\pm 1$	$\pm 3/2$	$\pm 2$
value	2	2	-1	-1	-1

The helicity content of  $256 \left| \sin \frac{\pi v}{2} \right|^8$  is

$\lambda_{\text{tot}}$	0	$\pm 1/2$	$\pm 1$	$\pm 3/2$	$\pm 2$
value	70	-56	28	-8	1

We see that with the normalization (145) we indeed reproduce the untwisted spectrum (123).

Let us also compute the second helicity supertrace

$$B_2 = - \left( \frac{1}{2\pi i} \partial_v - \frac{1}{2\pi i} \partial_{\bar{v}} \right)^2 Z(v, \bar{v}) \Big|_{v=\bar{v}=0}. \quad (146)$$

Using

$$\partial_\nu \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\nu, \tau) \Big|_{\nu=0} = -2\pi \eta(\tau)^3, \quad \frac{\partial}{\partial \nu} = \frac{1}{2} \frac{\partial}{\partial \frac{\nu}{2}}, \quad (147)$$

we find from (127) and (128)

$$B_2 = 36. \quad (148)$$

The massless contribution from (141) and (142) gives the same

$$B_2|_{\text{massless}} = 27 + 9 = 36. \quad (149)$$

The  $\mathbb{Z}_3$  orbifold is the singular limit of the Eguchi-Hanson space  $EH_3$  that is a Calabi-Yau 3-fold with  $h^{1,1} = 36$  and  $h^{2,1} = 0$ . We get from (7)

$$\Delta \mathcal{L}_{\text{eff}}^{1\text{-loop}} = \frac{6}{\pi} M_s^2 \sqrt{-g} R. \quad (150)$$

### B.1.3 The non-compact case

For the  $\mathbb{Z}_3$  orbifold we have in the non-compact case from the 27 fixed points of the compact case just the origin left. Instead of  $C = -3$  in (131) we have  $C = -1$ . For the second helicity supertrace we get

$$B_2^T = B_2^T|_{\text{massless}} = 1 \quad (151)$$

that gives using (7)

$$\Delta\mathcal{L}_{\text{eff}}^{1\text{-loop}} = \frac{1}{6\pi} M_s^2 \sqrt{-g} R. \quad (152)$$

## B.2 General $N$

We consider the non-compact case and fix the normalization from the twisted sectors. The massless spectrum is given by (24) or (25), where the sectors  $h$  and  $N - h$  (for  $h \neq \frac{N}{2}$ ) together give one vector multiplet as does the sector  $h = \frac{N}{2}$  if  $N$  is even.

We will proof that the normalization of the partition function is given by (23) in the case that  $N$  is prime. The proof in the general case  $N \in \mathbb{N}$  relies on the fact that every natural number can uniquely be written as a product of prime numbers and is quite lengthy as there are sectors with  $hv_i \in \mathbb{Z}$  and one has more cases to consider. However, this generalization is straight forward. We will also assume that  $v_i \notin \mathbb{Z}$ ,  $i = 1, 2, 3$ . The  $v_i$  are of the form  $v_i = \frac{k_i}{N}$  with  $k_i \in \mathbb{Z}$ . As  $h = 1, \dots, N - 1$  and  $N$  is prime it follows that  $hv_i \notin \mathbb{Z}$  for all  $h$ .

From the partition function (10) we get the massless contribution of the twisted sectors from the limit  $\tau_2 \rightarrow \infty$ . Using (134) to (136) and  $C^{(N)} = -1$  we arrive at

$$Z^T(v, \bar{v})|_{\text{massless}} = -N_0 4 \left| \sin \frac{\pi v}{2} \right|^2 \sum_{g=0}^{N-1} \sum_{h=1}^{N-1} \prod_{i=1}^3 \lim_{\tau_2 \rightarrow \infty} \left| \frac{\theta \left[ \begin{smallmatrix} 1/2 + hv_i \\ 1/2 + gv_i \end{smallmatrix} \right] (v/2, \tau)}{\theta \left[ \begin{smallmatrix} 1/2 + hv_i \\ 1/2 + gv_i \end{smallmatrix} \right] (0, \tau)} \right|^2. \quad (153)$$

We write  $hv_i = [hv_i] + r(hv_i)$  with integer part  $[hv_i]$  and rest  $r(hv_i) \in (0, 1)$ . From the product representation of the theta functions (96) we get

$$\begin{aligned} & \prod_{i=1}^3 \sum_{g=0}^{N-1} \sum_{h=1}^{N-1} \lim_{\tau_2 \rightarrow \infty} \left| \frac{\theta \left[ \begin{smallmatrix} 1/2 + hv_i \\ 1/2 + gv_i \end{smallmatrix} \right] (v/2, \tau)}{\theta \left[ \begin{smallmatrix} 1/2 + hv_i \\ 1/2 + gv_i \end{smallmatrix} \right] (0, \tau)} \right|^2 = \prod_{i=1}^3 \sum_{g=0}^{N-1} \sum_{h=1}^{N-1} \left| e^{i\pi v(-\frac{1}{2} + r(hv_i))} \right|^2 \\ & = N \sum_{h=1}^{N-1} \left| e^{-\frac{3}{2}i\pi v} e^{i\pi v h \sum_{i=1}^3 v_i} e^{-i\pi v \sum_{i=1}^3 [hv_i]} \right|^2. \end{aligned} \quad (154)$$

Due to supersymmetry we have  $v_3 = -v_1 - v_2$ . On the other hand  $[-x] = -[x] - 1$  for any  $x$  and  $[x_1 + x_2] = \begin{cases} [x_1] + [x_2] & \text{for } r(x_1) + r(x_2) < 1 \\ [x_1] + [x_2] + 1 & \text{for } r(x_1) + r(x_2) > 1 \end{cases}$  for any  $x_1$  and  $x_2$ . If  $r(hv_1) + r(hv_2) < 1$  ( $> 1$ ) then  $r((N-h)v_1) + r((N-h)v_2) = 1 - r(hv_1) + 1 - r(hv_2) > 1$  ( $< 1$ ). We are left with

$$Z^T(v, \bar{v})|_{\text{massless}} = -N_0 \, 4 \left| \sin \frac{\pi v}{2} \right|^2 N \frac{N-1}{2} \left( \left| e^{-\frac{i\pi v}{2}} \right|^2 + \left| e^{\frac{i\pi v}{2}} \right|^2 \right) \quad (155)$$

and from the helicity content of the functions after equation (144) follows the normalization (23) if one matches on the spectrum (25).

## C Some details of the two graviton amplitude

The part of the general n-point one-loop amplitude coming from even-even spin structures is given by (see [15])

$$\begin{aligned} \mathcal{A}_n^{(e,e)} &= \sum_{(\alpha,\beta)=0,1}^{\text{even}} \sum_{(\bar{\alpha},\bar{\beta})=0,1}^{\text{even}} \int \frac{d^2\tau}{\tau_2^2} (-)^{\alpha+\beta+\alpha\beta} (-)^{\bar{\alpha}+\bar{\beta}+\bar{\alpha}\bar{\beta}} Z(\tau, \bar{\tau}, (\alpha, \beta), (\bar{\alpha}, \bar{\beta})) \\ &\times \int_{\Gamma_\tau} \prod_{i=1}^{n-1} d^2 z_i \langle \prod_{i=1}^n V^{(0,0)}(z_i, \bar{z}_i) \rangle_{(\alpha,\beta), (\bar{\alpha},\bar{\beta})} \end{aligned} \quad (156)$$

as all vertex operators can be chosen in the  $(0,0)$ -ghost picture (see e.g. [29]),  $\Gamma$  is the fundamental region that is

$$\Gamma = \{ \tau | \text{Im}\tau > 0, |\text{Re}\tau| \leq \frac{1}{2}, |\tau| \geq 1 \} \quad (157)$$

for the torus and  $\tau_2 \in [0, \infty]$  for  $\mathcal{K}, \mathcal{A}, \mathcal{M}$  and the  $z_i$  are integrated over the strip

$$\Gamma_\tau = \{ z_i | |\text{Re}z_i| \leq \frac{1}{2}, 0 \leq \text{Im}z_i \leq \text{Im}\tau \}. \quad (158)$$

For the torus we can set  $z_n = \tau$  due to the conformal symmetry. The partition function is vanishing by supersymmetry so we need at least two fermion contractions (from the vertex operators) to get a non-vanishing result. The graviton vertex operator in the  $(0,0)$ -ghost picture is

$$V^{(0,0)}(z, \bar{z}) = -\frac{2g_s}{\alpha'} \varepsilon_{\mu\nu} : \left( i\partial X^\mu - \frac{\alpha'}{2} \psi^\mu p \cdot \psi \right) \left( i\bar{\partial} X^\nu + \frac{\alpha'}{2} \tilde{\psi}^\nu p \cdot \tilde{\psi} \right) e^{ip \cdot X} : . \quad (159)$$

The bosonic Green function on the torus is

$$\langle X^\mu(z, \bar{z}) X^\nu(z', \bar{z}') \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \log |\chi(z - z', \tau)|^2, \quad (160)$$

where

$$\chi(z_{ij}, \tau) = 2\pi \exp \left[ -\pi \frac{(\text{Im} z_{ij})^2}{\text{Im} \tau} \right] \frac{\theta \left[ \frac{1/2}{1/2} \right] (z_{ij}, \tau)}{\theta' \left[ \frac{1/2}{1/2} \right] (0, \tau)}. \quad (161)$$

The fermionic Green function on the torus is

$$\langle \psi^\mu(z) \psi^\nu(z') \rangle_{(\alpha, \beta)} = \eta^{\mu\nu} \frac{\theta \left[ \frac{\alpha/2}{\beta/2} \right] (z - z', \tau) \theta' \left[ \frac{1/2}{1/2} \right] (0, \tau)}{\theta \left[ \frac{1/2}{1/2} \right] (z - z', \tau) \theta \left[ \frac{\alpha/2}{\beta/2} \right] (0, \tau)}. \quad (162)$$

For the bosonic correlation functions we use as a starting point

$$\begin{aligned} & \langle \exp \left[ \sum_{i=1}^N \left( ik_i \cdot X(z_i, \bar{z}_i) + J^\mu \partial_{z_i} X_\mu(z_i, \bar{z}_i) + \bar{J}^\mu \partial_{\bar{z}_i} X_\mu(z_i, \bar{z}_i) \right) \right] \rangle = \\ & = \exp \left[ \frac{1}{2} \sum_{i \neq j}^N \left( ik_{i\mu} + J_\mu(z_i) \partial_{z_i} + \bar{J}_\mu(\bar{z}_i) \partial_{\bar{z}_i} \right) \left( ik_{j\nu} + J_\nu(z_j) \partial_{z_j} + \bar{J}_\nu(\bar{z}_j) \partial_{\bar{z}_j} \right) \right. \\ & \quad \left. \times \langle X^\mu(z_i, \bar{z}_i) X^\nu(z_j, \bar{z}_j) \rangle \right]. \end{aligned} \quad (163)$$

Making functional derivatives with respect to the currents  $J(z)$  and  $\bar{J}(\bar{z})$  and finally setting them to zero, we can compute the expectation value of any vertex operator that is a polynomial in derivatives of  $X$  times the exponential  $e^{ik \cdot X}$ . We define

$$G(z, \tau) = -\frac{1}{2} \log |\chi(z, \tau)| \quad (164)$$

and find

$$\begin{aligned} \partial_z G(z, \tau) &= -\frac{1}{4} \sum'_{k,m} \frac{1}{k\tau - m} \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \tau \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\ \partial_{\bar{z}} G(z, \tau) &= \frac{1}{4} \sum'_{k,m} \frac{1}{k\bar{\tau} - m} \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \tau \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\ \partial_z \partial_z G(z, \tau) &= \frac{\pi}{4\text{Im} \tau} \sum'_{k,m} \frac{m - k\bar{\tau}}{m - k\tau} \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \tau \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\ \partial_{\bar{z}} \partial_{\bar{z}} G(z, \tau) &= \frac{\pi}{4\text{Im} \tau} \sum'_{k,m} \frac{m - k\tau}{m - k\bar{\tau}} \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \tau \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\ \partial_z \partial_{\bar{z}} G(z, \tau) &= -\frac{\pi}{4\text{Im} \tau} \sum'_{k,m} \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \tau \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\ &= -\frac{\pi}{4} \left( \delta(\text{Re} z) \delta(\text{Im} z) - \frac{1}{\text{Im} \tau} \right), \end{aligned} \quad (165)$$

where  $\sum_{k,m}'$  means that  $(k, m) = (0, 0)$  is not in the sum. This gives

$$\int_0^{\text{Im}\tau} d\text{Im}z \int_{-\frac{1}{2}}^{\frac{1}{2}} d\text{Re}z \partial_z \partial_z G(z, \tau) = 0 \quad (166)$$

$$\int_0^{\text{Im}\tau} d\text{Im}z \int_{-\frac{1}{2}}^{\frac{1}{2}} d\text{Re}z \partial_z \partial_{\bar{z}} G(z, \tau) = 0. \quad (167)$$

## D Deriving the tadpole conditions

First we find the transverse channel expressions for the amplitudes. For the annulus and Klein bottle this is achieved by the standard  $S$ -transformation as they depend on  $\frac{1}{2}i\tau_2$  and  $2i\tau_2$  respectively. For the Moebius amplitude the functions do not depend on the standard  $\frac{1}{2} + \frac{1}{2}i\tau_2$  that would lead to a  $P$ -transformation (with  $P = ST^2ST$ ) but on  $i\tau_2$  (because they depend on  $q_{\text{new}} = q_{\text{old}}^2$  see (55)) so we again need a  $S$ -transformation to go to the transverse channel. The  $S$ -transformation of the theta functions is given by (92).

For the annulus we have the transformations  $t = \frac{1}{2}\tau_2, l = \frac{1}{t}$

$$\begin{aligned} Z_{\mathcal{A}} = & \frac{1}{2^2} \frac{(1-1)}{4N} \int_0^\infty dl \sum_{k=0}^{N-1} \frac{\theta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0, il)}{\eta(il)^3} \\ & \times \prod_{i=1}^3 |2 \sin(\pi k v_i)| \frac{\theta \begin{bmatrix} 1/2 + k v_i \\ 0 \end{bmatrix} (0, il)}{(-i) e^{-i\pi k v_i} \theta \begin{bmatrix} 1/2 + k v_i \\ 1/2 \end{bmatrix} (0, il)} (\text{Tr } \gamma_{k,3})^2. \end{aligned} \quad (168)$$

For the Moebius strip we have the transformations  $t = \frac{1}{\tau_2}, l = \frac{t}{2}$

$$\begin{aligned} Z_{\mathcal{M}} = & 2 \frac{(1-1)}{4N} \int_0^\infty dl \sum_{k=0}^{N-1} \frac{\theta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0, 2il) \theta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0, 2il)}{\eta(2il)^3 \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2il)} \\ & \times \prod_{i=1}^3 s_i(-2 \sin(\pi k v_i)) \frac{e^{-i\pi k v_i} \theta \begin{bmatrix} k v_i \\ 1/2 \end{bmatrix} (0, 2il) \theta \begin{bmatrix} 1/2 + k v_i \\ 0 \end{bmatrix} (0, 2il)}{(-i) e^{-i\pi k v_i} \theta \begin{bmatrix} 1/2 + k v_i \\ 1/2 \end{bmatrix} (0, 2il) \theta \begin{bmatrix} k v_i \\ 0 \end{bmatrix} (0, 2il)} \\ & \times \text{Tr } \gamma_{\Omega_k,3}^{-1} \gamma_{\Omega_k,3}^T. \end{aligned} \quad (169)$$

For the Klein bottle we have the transformations  $t = 2\tau_2, l = \frac{1}{t}$

$$\begin{aligned}
Z_{\mathcal{K}} = & 2^2 \frac{(1-1)}{4N} \int_0^\infty dl \sum_{k=0}^{N-1} \frac{\theta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0, il)}{\eta(il)^3} \\
& \times \prod_{i=1}^3 \frac{|2 \sin(2\pi k v_i)|}{4(\sin(\pi(k v_i + \frac{1}{2})))^2} \frac{\theta \begin{bmatrix} 1/2 + 2k v_i \\ 0 \end{bmatrix} (0, il)}{(-i)e^{-2i\pi k v_i} \theta \begin{bmatrix} 1/2 + 2k v_i \\ 1/2 \end{bmatrix} (0, il)}.
\end{aligned} \tag{170}$$

The ultraviolet contribution comes from  $\tau_2 \rightarrow 0$  or  $l \rightarrow \infty$ . We have

$$\lim_{l \rightarrow \infty} \frac{\theta \begin{bmatrix} a \\ b_1 \end{bmatrix} (0, il)}{\theta \begin{bmatrix} a \\ b_2 \end{bmatrix} (0, il)} = e^{2\pi i a(b_1 - b_2)}. \tag{171}$$

In the sum  $Z_{\mathcal{A}} + Z_{\mathcal{M}} + Z_{\mathcal{K}}$  the NS and R contributions are both separately free of ultraviolet divergences, i.e. of tadpoles, under the condition that

$$\begin{aligned}
0 = & \frac{1}{4} \prod_{i=1}^3 |2 \sin(\pi k v_i)| (\text{Tr } \gamma_{k,3})^2 + 2 \prod_{i=1}^3 s_i (-2 \sin(\pi k v_i)) \text{Tr } (\gamma_{\Omega_k,3}^{-1} \gamma_{\Omega_k,3}^T) \\
& + 4 \prod_{i=1}^3 \frac{|2 \sin(2\pi k v_i)|}{4(\sin(\pi(k v_i + \frac{1}{2})))^2}.
\end{aligned} \tag{172}$$

We can choose  $\text{Tr } (\gamma_{\Omega_k,3}^{-1} \gamma_{\Omega_k,3}^T) = \pm \text{Tr } \gamma_{2k,3}$ , where the positive sign is the  $SO$  projection and the negative sign is the  $Sp$  projection. We have

$$\frac{1}{4} \prod_{i=1}^3 |2 \sin(\pi k v_i)| (\text{Tr } \gamma_{k,3})^2 = \frac{1}{4} \prod_{i=1}^3 |2 \sin(2\pi k v_i)| (\text{Tr } \gamma_{2k,3})^2. \tag{173}$$

Using  $\sin(2\pi k v_i) = 2 \sin(\pi k v_i) \cos(\pi k v_i)$  and  $\sin(\pi(k v_i + \frac{1}{2})) = \cos(\pi k v_i)$  we find that (172) is equivalent to

$$0 = \frac{1}{4} \left[ \prod_{i=1}^3 |2 \sin(2\pi k v_i)| \right] \left( \text{Tr } \gamma_{2k,3} \mp 4 \prod_{i=1}^3 \frac{1}{2 \cos(\pi k v_i)} \right)^2 \tag{174}$$

and is a perfect square.

## References

- [1] E. Kiritsis, [arXiv:hep-th/0310001].

- [2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 [arXiv:hep-ph/9905221].
- [3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690 [arXiv:hep-th/9906064].
- [4] G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. **B484** (2000) 112 [arXiv:hep-th/0002190].
- [5] G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. **B485** (2000) 208 [arXiv:hep-th/0005016].
- [6] G. Dvali and G. Gabadadze, Phys. Rev. **D63** (2001) 065007 [arXiv:hep-th/0008054].
- [7] G. Dvali, G. Gabadadze, M. Kolanovic and F. Nitti, Phys. Rev. **D64** (2001) 084004 [arXiv:hep-ph/0102216].
- [8] E. Kiritsis, N. Tetradis and T. Tomaras, JHEP **0108** (2001) 012 [arXiv:hep-th/0106050].
- [9] C. Middleton and G. Siopsis, [arXiv:hep-th/0210033].
- [10] U. Ellwanger, [arXiv:hep-th/0304057].
- [11] M. Kolanovic, Phys. Rev. **D67** (2003) 106002 [arXiv:hep-th/0301116].
- [12] M. Kolanovic, M. Porrati and J.-W. Rombouts, Phys. Rev. **D68** (2003) 064018, [arXiv:hep-th/0304148].
- [13] E. Kiritsis, N. Tetradis and T. Tomaras, JHEP **0203** (2002) 019 [arXiv:hep-th/0202037].
- [14] E. Kiritsis and C. Kounnas, Nucl. Phys. **B442** (1995) 472 [arXiv:hep-th/9501020].
- [15] K. Förger, B.A. Ovrut, S.J. Theisen and D. Waldram, Phys. Lett. **B388** (1996) 512 [arXiv:hep-th/9605145].
- [16] E. Kiritsis, Leuven Univ. Press (1998) 315 p, (Leuven notes in mathematical and theoretical physics. B9), [arXiv:hep-th/9709062].
- [17] I. Antoniadis, S. Ferrara, R. Minasian and K.S. Narain, Nucl. Phys. **B507** (1997) 571 [arXiv:hep-th/9707013].
- [18] E. Kohlprath, JHEP **0210** (2002) 026 [arXiv:hep-th/0207023].
- [19] I. Antoniadis, R. Minasian and P. Vanhove, Nucl. Phys. **B648** (2003) 69 [arXiv:hep-th/0209030].

- [20] Z. Kakushadze, Nucl. Phys. **B529** (1998) 157 [arXiv:hep-th/9803214].
- [21] Z. Kakushadze, JHEP **0110** (2001) 031 [arXiv:hep-th/0109054].
- [22] L.E. Ibanez, R. Rabadan and A.M. Uranga, Nucl. Phys. **B542** (1999) 112 [arXiv:hep-th/9808139].
- [23] I. Antoniadis, C. Bachas, C. Fabre, H. Partouche and T.R. Taylor, Nucl. Phys. **B489** (1997) 160 [arXiv:hep-th/9608012].
- [24] I. Antoniadis, R. Minasian, S. Theisen and P. Vanhove, Class. Quant. Grav. **20** (2003) 5079 [arXiv:hep-th/0307268].
- [25] G. Aldazabal, A. Font, L.E. Ibanez und G. Violero, Nucl. Phys. **B536** (1998) 29 [arXiv:hep-th/9804026].
- [26] E.C. Gimon und J. Polchinski, Phys. Rev. **D54** (1996) 1667 [arXiv:hep-th/9601038].
- [27] C. Angelantonj und A. Sagnotti, Phys. Rept. **371** (2002) 1, Erratum-ibid. **376** (2003) 339 [arXiv:hep-th/0204089].
- [28] J. Polchinski, *String Theory Volumes I and II*, Cambridge University Press, 1998.
- [29] W. Lerche, B.E.W. Nilsson and A.N. Schellekens, Nucl. Phys. **B289** (1987) 609.